

ON THE REPRESENTATION OF INTEGERS BY BINARY QUADRATIC FORMS

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ABSTRACT. In this note we show that for a given irreducible binary quadratic form $f(x, y)$ with integer coefficients, whenever we have $f(x, y) = f(u, v)$ for integers x, y, u, v , there exists a rational automorphism of f which sends (x, y) to (u, v) .

1. INTRODUCTION

Let F be a binary form with integer coefficients, non-zero discriminant, and degree $d \geq 2$. We say that an integer h is *representable* by F if there exist integers x, y such that $F(x, y) = h$. It is an old question, dating back to Diophantus in the case of sums of two squares, to determine which integers h are representable by a given form F . While an exact description (for example, in terms of congruence conditions) remain elusive for all but the simplest of cases, asymptotic results have now been established. Define

$$(1.1) \quad \mathcal{R}_F(Z) = \{h \in \mathbb{Z} : h \text{ is representable by } F, |h| \leq Z\}$$

and $R_F(Z) = \#\mathcal{R}_F(Z)$. Landau proved in 1908 that there exists a positive number C_1 such that

$$(1.2) \quad R_{x^2+y^2}(Z) \sim \frac{C_1 Z}{\sqrt{\log Z}},$$

and shortly after the result was established for all positive definite binary quadratic forms.

In general, one expects that for a binary form F with degree $d \geq 3$, integer coefficients, and non-zero discriminant, that there exists a positive number $C(F)$ such that the asymptotic relation

$$(1.3) \quad R_F(Z) \sim C(F)Z^{\frac{2}{d}}$$

holds. It would take over half a century before the analogous asymptotic formula would be established for non-abelian cubic forms, which was achieved by Hooley. He proved in [3] that (1.3) holds whenever F is a non-abelian binary cubic form. In subsequent works [4] [5], he established (1.3) for bi-quadratic binary quartic forms and abelian binary cubic forms, respectively. In [7], Stewart and Xiao established (1.3) for all integral binary forms of degree $d \geq 3$ and non-zero discriminant.

For F a binary form of degree $d \geq 2$, define

$$\text{Aut}_{\mathbb{Q}} F = \left\{ T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \in \text{GL}_2(\mathbb{Q}) : F(x, y) = F(t_1x + t_2y, t_3x + t_4y) \right\}.$$

The absence of the logarithmic term in (1.3) as opposed to (1.2) is accounted for by the fact that for a binary form F of degree at least 3, $\text{Aut}_{\mathbb{Q}} F$ is always finite. When

F is a quadratic form, the group $\text{Aut}_{\mathbb{Q}} F$ is infinite.

We say a representable integer h is *essentially represented* if whenever $(x, y), (u, v) \in \mathbb{Z}^2$ are such that $F(x, y) = F(u, v) = h$, there exists $T \in \text{Aut}_{\mathbb{Q}} F$ such that $\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}$. Note that if $F(x, y) = h$ has a unique solution, then h is essentially represented since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Aut}_{\mathbb{Q}} F$. Put

$$\mathcal{R}_F^{(1)}(Z) = \{h \in \mathcal{R}_F(Z) : h \text{ is essentially represented}\}$$

and $R_F^{(1)}(Z) = \#\mathcal{R}_F^{(1)}(Z)$. In the $d \geq 3$ case Heath-Brown showed in [2] that there exists $\eta_d > 0$, depending only on the degree d , such that for all $\varepsilon > 0$

$$(1.4) \quad R_F(Z) = R_F^{(1)}(Z) \left(1 + O_{\varepsilon}(Z^{-\eta_d + \varepsilon})\right).$$

This essentially reduces the question of enumerating $\mathcal{R}_F(Z)$ to that of $\mathcal{R}_F^{(1)}(Z)$, which is far simpler, and the key to our success in [7]. Heath-Brown's theorem does not address the case of quadratic forms, which we do so now:

Theorem 1.1. *Let f be an irreducible and primitive binary quadratic form. Then every integer h representable by f is essentially represented.*

Consider the quadric surface X_f defined by

$$X_f : f(x_1, x_2) = f(x_3, x_4).$$

In [2], Heath-Brown showed that lines on X_f correspond to automorphisms of f , possibly defined over a larger field. His result and our Theorem 1.1 has the following consequence:

Corollary 1.2. *Let X_f be the surface defined by $f(x_1, x_2) = f(x_3, x_4)$, with f a binary quadratic form with integer coefficients and non-zero discriminant. Then every point in $X_f(\mathbb{Q})$ lies on a rational line contained in X_f .*

It has been pointed out to the author that Theorem 1.1 essentially follows from Witt's theorem (see Theorem 42.16 in [1]). Nevertheless, we feel that this result is of independent interest to number theorists and does not appear to be well-known.

2. PRELIMINARY LEMMAS

The strategy is very simple: for a given pair of integers $(x, y), (u, v)$ such that $f(x, y) = f(u, v)$, we exhibit an explicit automorphism of f which sends (x, y) to (u, v) . In fact, we will draw such an automorphism from a proper subgroup of $\text{Aut}_{\mathbb{Q}} f$. Put

$$f(x, y) = f_2 x^2 + f_1 xy + f_0 y^2,$$

and put

$$\delta = \left| \frac{f_1^2 - 4f_2 f_0}{4} \right|.$$

We shall first characterize the automorphism group $\text{Aut}_{\mathbb{Q}} f$. It turns out that this depends on whether f is positive definite or not.

2.1. Positive definite binary quadratic forms. In this case, we shall pick our T from the group $\text{Aut}_{\mathbb{Q}} f \cap SO_f(\mathbb{R})$, where

$$SO_f(\mathbb{R}) = \{T \in \text{GL}_2(\mathbb{R}) : \det T = 1, f_T = f\}.$$

The group $SO_f(\mathbb{R})$ is conjugate to the special orthogonal group $SO_2(\mathbb{R})$ and its elements look like

$$T_f(t) = \begin{pmatrix} \cos t + \frac{f_1 \sin t}{2\sqrt{\delta}} & \frac{f_0 \sin t}{\sqrt{\delta}} \\ \frac{-f_2 \sin t}{\sqrt{\delta}} & \cos t - \frac{f_1 \sin t}{2\sqrt{\delta}} \end{pmatrix}, t \in [0, 2\pi).$$

If we demand that $T_f(t) \in \text{GL}_2(\mathbb{Q})$, then it follows that $\cos t \in \mathbb{Q}$ and $\sqrt{\delta} \sin t \in \mathbb{Q}$. Put

$$u = \cos t, v = \frac{\sin t}{\sqrt{\delta}}.$$

Then u, v satisfy the equation

$$u^2 + \delta v^2 = 1.$$

Put E_{δ} for the curve defined by

$$(2.1) \quad E_{\delta} : x^2 + \delta y^2 = 1.$$

We then see that there is a bijection between rational points on E_{δ} and rational elements $T \in \text{Aut}_{\mathbb{Q}} f$. We now characterize the set of rational points on E_{δ} .

Lemma 2.1. *Let E_{δ} be the curve given by (2.1), with 4δ a positive integer. Then the set of rational points on E_{δ} is given by the parametrization*

$$\left(\frac{\delta p^2 - q^2}{\delta p^2 + q^2}, \frac{2pq}{\delta p^2 + q^2} \right), p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1.$$

Proof. Using the fact that $(1, 0)$ is a point on the curve E_{δ} , we use the slope method to find all other rational points. Indeed, the intersection of the line given by

$$y = m(x - 1), m \in \mathbb{Q}$$

and the curve E_{δ} is another rational point on E_{δ} , and all such points arise this way. Substituting, we find that

$$x^2 + \delta(m(x - 1))^2 = 1$$

is equivalent to

$$x = \frac{\delta m^2 \pm 1}{\delta m^2 + 1}.$$

The $+$ sign gives $x = 1$, and the $-$ sign gives

$$x = \frac{\delta m^2 - 1}{\delta m^2 + 1}$$

which corresponds to the point

$$(x, y) = \left(\frac{\delta m^2 - 1}{\delta m^2 + 1}, \frac{2m}{\delta m^2 + 1} \right).$$

If we write the slope m as $m = p/q$, where $q > 0$ and $\gcd(p, q) = 1$, then the point can be given as

$$(x, y) = \left(\frac{\delta p^2 - q^2}{\delta p^2 + q^2}, \frac{2pq}{\delta p^2 + q^2} \right),$$

as desired. \square

2.2. Indefinite binary quadratic forms. In this case, the group $SO_f(\mathbb{R})$ is no longer connected, and we shall focus on the *principal branch* of $SO_f(\mathbb{R})$, which is the branch containing the identity matrix. This branch can be identified as the set of matrices of the form

$$T_f(t) = \begin{pmatrix} \cosh t - \frac{f_1 \sinh t}{2\sqrt{\delta}} & -\frac{f_0 \sinh t}{\sqrt{\delta}} \\ \frac{f_2 \sinh t}{\sqrt{\delta}} & \cosh t + \frac{f_1 \sinh t}{2\sqrt{\delta}} \end{pmatrix}, t \in \mathbb{R}.$$

Again, if we demand that $T_f(t) \in \mathrm{GL}_2(\mathbb{Q})$, then necessarily $\cosh t, \sqrt{\delta} \sinh t \in \mathbb{Q}$. Put

$$u = \cosh t, v = \frac{\sinh t}{\sqrt{\delta}}.$$

Notice that (u, v) lies on the curve

$$(2.2) \quad E_\delta : x^2 - \delta y^2 = 1.$$

It is immediate that there is a bijection between the set of rational points $E_\delta(\mathbb{Q})$ and elements in $SO_f(\mathbb{Q})$. We have the following characterization of the rational points on E_δ :

Lemma 2.2. *Let E_δ be the curve given by (2.2). Then the set of rational points $E_\delta(\mathbb{Q})$ are given by the parametrization*

$$\left(\frac{\delta p^2 + q^2}{\delta p^2 - q^2}, \frac{2pq}{\delta p^2 - q^2} \right), p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1.$$

Proof. Same as Lemma 2.1. \square

3. PROOF OF THEOREM 1.1

We first address the case when f is positive definite. Let h be a representable integer of f . If there exists exactly one pair of integers (x, y) such that $f(x, y) = h$, then h is essentially represented. Now suppose there exist distinct representations $(x, y), (u, v)$ of h , so that

$$(3.1) \quad h = f(x, y) = f(u, v).$$

Put

$$m = 2f_2ux + f_1(uy + vx) + 2f_0vy - 2h, n = 2\delta(uy - vx)$$

and

$$T_f(m, n) = \frac{1}{\delta m^2 + n^2} \begin{pmatrix} \delta m^2 - n^2 + f_1mn & 2f_0mn \\ -2f_2mn & \delta m^2 - n^2 - f_1mn \end{pmatrix} \in \mathrm{Aut}_{\mathbb{Q}} f.$$

Observe that

$$(\delta m^2 - n^2 + f_1mn)x + 2f_0mny = hm\delta u$$

and

$$-2f_2mnx + (\delta m^2 - n^2 - f_1mn)y = hm\delta v.$$

Moreover, by expanding, we see that

$$\delta m^2 + n^2 = hm\delta.$$

It then follows that

$$T_f(m, n) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

The proof of the theorem when f is indefinite is similar, but we include the full argument for the sake of completeness. Suppose that (3.1) holds and put

$$m = 2f_2ux + f_1(uy + vx) + 2f_0vy - 2h, n = 2\delta(vx - uy).$$

Then the associated $T_f(m, n) \in \text{Aut}_{\mathbb{Q}} f$ is given by

$$T_f(m, n) = \frac{1}{\delta m^2 - n^2} \begin{pmatrix} \delta m^2 + n^2 - f_1mn & -2f_0mn \\ 2f_2mn & \delta m^2 + n^2 + f_1mn \end{pmatrix}.$$

A routine calculation then yields that

$$T_f(m, n) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

as desired.

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